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Four-dimensional Galois representations and paramodular forms

Calabi–Yau Galois representations: computations, modularity, congruences. A search for Laurent polynomials f in dimension 4 that give rise to 4–th order DEs enables one to approach certain aspects of the Langlands correspondence for $GSp(2)$ in a direct and explicit way. Recall that for an elliptic curve over the rationals, its L–function is defined as the Euler product $L(E,s) = \prod_{\text{bad }p}(1-a_p p^{-s})^{-1}\prod_{\text{good }p}(1-a_p p^{-s}+p^{1-2s})^{-1}$, where the Euler factor for a prime p of good reduction is $\prod_{i=0}^{1}(1-\alpha_i p^{-s})^{-1}$, $\alpha_{0,1}$ being the eigenvalues of the p–Frobenius acting in $H^1_{\text{ét}}(\overline{E},\mathbb{Q}_l)$. By the famous Taniyama–Weil conjecture, or the BCDT theorem, the complete L–function $L^*(E,s) = \pi^{-s}N^{s/2}\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2})L(E,s)$ with $N = N_E$=conductor of E extends to an analytic function in the entire complex plane and satisfies the functional equation $L^*(E,s) = \pm L^*(E,2-s)$. This statement is a direct consequence of the modularity of the q–series whose Mellin transform is L.

It has been expected, since Langlands, that other motivic L–functions are automorphic (i.e. can be associated to automorphic representations) as well and can therefore be analytically continued and satisfy functional equations of a similar shape. The case that lends itself to consideration next after elliptic curves is that of weight 3 four–dimensional Galois representations R of Calabi-Yau type. Defining the Euler factor of $L(R,s)$ for p of good reduction to be $\prod_{i=0}^{3}(1-\beta_i p^{-s})^{-1}$ with $\beta_{0,1,2,3}$ the eigenvalues of the p–Frobenius, and fixing the product by finitely many bad factors determined by the inertia action in a standard way, we ask whether $L^*(R,s) = \pi^{-2\,s}N_R^{s/2}\Gamma(\frac{s-1}{2})\Gamma(\frac{s}{2})^2\Gamma(\frac{s+1}{2})L(R,s)$ can be analytically continued to an entire function satisfying $L^*(R,s) = \pm L^*(R,4-s)$. Getting a grip on these L's poses a challenge: in contrast to the motivic weight 1 situation (which should be thought of as the prototypical CY case with $h^{10} = h^{01} = 1$) where a rank 2 Galois representation can be realized geometrically as arising in the $H^1_{\text{ét}}$ of some elliptic curve simply by Weierstrass, we do not know of a similar way to realize weight 3 CY Galois representations with $h^{30} = h^{21} = h^{12} = h^{03} = 1$ in the H^3 of a Calabi-Yau type variety via any kind of a uniform construction! In the absence of this, the technology leans naturally toward our methods: one systematic way to catalog rank 4 symplectic representations à la Cremona is to study the relevant pieces in the middle cohomology of the hypersurfaces $E_t : f = t$ of our Laurent polynomials in an intelligent manner. The p-adic methods originating in Dwork's work use the classical period/DE directly, making this approach much more cost-efficient than the naive point count followed by stripping away the contributions from the parasitic constituents of $H^3(E_t)$.

More conceptually, given such a 4-dimensional Galois representation obtained, say, as a collection of the Frobenius eigenvalues for each F_p, we shall try, in the framework of the Langlands philosophy, to find an automorphic object, typically a paramodular form, that corresponds to it, thereby linking our study to work of Gritsenko, Poor, Weissauer and Yuen. Independently of that, one can always study its L–function for the standard analytic predictions (analytic continuation/functional equation) numerically.

An exciting arithmetic prediction about the Galois representations arising in the level hypersurfaces of the LG models of Fano varieties is that their congruence properties [in the sense of Serre's explanation of the famous Ramanujan mod 691 congruence] should reflect the numerical invariants of the respective Fanos. In particular, the anticanonical degree is expected to appear in many cases as the Bloch-Kato factor of certain critical L-values. A prototypical example was worked out very recently by Dummigan and Golyshev for the variety V_{22} in dimension 3. Perhaps even more exciting is the existence of Galois reps possessing *double* congruences. In our notation, this means, roughly, the existence of two congruence moduli N_1, N_2 such that $\beta_1 = p, \beta_2 = p^2 \mod N_1$ and $\beta_0 = 1, \beta_3 = p^3 \mod N_2$ for almost every prime p. Predicted by Golyshev, the first example of the "second congruence" phenomenon was found numerically by Buzzard with $N_R = 61$, $N_1 = 43$, $N_2 = 19$ (the first congruence had been discovered earlier by Poor and Yuen in this case).

Objectives. General automorphy theorems for $GSp(2)$, being a highly techhnical and specialized subject, are per se clearly beyond the scope of our study. Still, as the main outcome of the first year, we expect to have amassed a large database of rank 4 motivic L–functions of low conductors (a '$GSp(2)$ Cremona table') which could be analyzed individually according to the standard methodology of arithmetic geometry; in fact, what Swinnerton–Dyer did many decades ago for elliptic curves makes perfect sense in the $GSp(2)$ case today. We will study motivic pieces of the Siegel modular threefolds and consistently try to associate those, and paramodular forms, to the known L–functions. We will study special values of these L–functions, and specifically $L(2)$, looking for the numerical evidence in support of Deligne's conjecture on critical values. We will study the congruence properties of the L–functions and compute numerically some of the Bloch–Kato factors, verifying at least some part of the Bloch–Kato predictions numerically. We will try to relate the congruence moduli to the motivic torsion in the Siegel threefolds, on the one hand, and to the characteristic classes of the Fano varieties whose Landau–Ginsburg models contain our CY motives in the fibers, thereby relating the Ramanujan–Harder phenomenon to topology in a not–quite–expected way.

Noriko: **(Para)modularity expectations in more detail.** This is a proposal about the modularity/automorphy of four-dimensional Galois representations arising from Calabi-Yau threefolds over Q with all the Hodge numbers of the third cohomology gorups equal to 1. This gives rise to four-dimensional Galois representations associated to such Calabi–Yau threefolds. The L-series of such Calabi-Yau threefolds may be computed in principle by finding local Euler factors for good primes, which boils down to counting numbers of rational points over finite fields.

There are many Calabi-Yau threefolds with the above property. Some are realized as mirror one-parameter families of Calabi-Yau threefolds. Hypersurfaces, complete intersections of hypersurfaces in weighted projective spaces, or toric constructions would give explicit defining equations. First, we would like to have a huge data basis for these Calabi-Yau threefolds. Then for each Calabi-Yau threefold, we wish to determine the conductor (i.e., product of powers of bad primes). Finally, we try to compute the Euler factors for each good prime. The Euler factors are derived from degree 4 polynomials with integer coefficients. We assume that these degree 4 polynomials are irreducible over Q.

In order to address the Langlands correspondence, we need to have some modular objects. In our case, we will consider paramodular subgroups of Sp(4,Z). This is motivated by a paramodular conjecture for abelian surfaces by Brumer and Kramer. Their conjecture is that every abelian surface A over Q with condoctir N and End(A)=Z should be paramodular, that is, there eixsts a paramodular Siegel modular form F of weight 2 and genus 2 with rational eigenvalues which determine the L-series of A. In our case, we make a very crude conjecture

Conjecture: Every Calabi-Yau threefold X over Q with all Hodge number of the third cohomology group equal to 1 is paramodular, that is, there exists a Siegel modular form of weight 3, genus 2 with conductor N for some subgroup of paramodular group, which determine the L-series of X.

Our goal is to supply examples (or counter-examples) in support of this conjecture.

Duco:

Non-trivial pencils of Calabi-Yau threefolds give rise to motivic local systems with $h^{30} = h^{21} = h^{12} = h^{03} = 1$ and thus one-parameter families of 4-dimensional Galois-representations and L-functions. Conjectures of Golyshev aim to characterise the common feature of these one-parameter families of 4-dimensional Galois-representations in terms of bi-congruences and seek a modular interpreteation of such pencils. Apart from the pencils arising from Laurent polynomials in dimension four, there are various other sources of examples. An attractive class are the fibre products of rational elliptic surfaces, introduced by C. Schoen. The cases leading to 4-dimensional H^3 were studied by Cynk and van Straten. For these spaces an LG-realisation as Laurent-polynomial is often not known. These examples give rise to Calabi-Yau operators: fourth order DE-operators which are amendable to the Dwork-method for computing the Euler-factors for fibres of the family. In recent years, this method was explored by Candelas, de la Ossa and van Straten, who extended the scope of the Dwork-method by starting at a point of maximal degeneration. It turns out that the initial data needed are determined by mirror Fano-data and p-adic $\zeta(3)$, thus giving a glimps of conjectural p-adic Gamma-conjectures. Using this input, L-factors can be computed quickly for virtually all Calabi-Yau operators. Using this method, versions of the the bi-congruences predicted by Golyshev were found recently by Candelas, de la Ossa and van Straten. Another class of examples is provided by special double octics, for which there are many examples where the corresponding DE does not have a point of maximal degeneration, the so-called orphans. For these the computation of the Euler-factors of the L-function from the DE is not straightforward. The algebra-geometrical study of the simplest Siegel-modular 3-folds by van Geemen, Nygaard, van Straten and Cynk, Freitag, Salvatti-Manni lead to many special Calabi-Yau 3-folds with $h^{30} = h^{21} = h^{12} = h^{03} = 1$. So these are special members of 1-parameter families and it would be very interesting to obtain explicit descriptions of these families.

Rainer: One of the missing links to arithmetic in the study of modular forms on $GSp(4)$ comes from the conjectured, but unexplained relationship between Siegel modular forms and algebraic modular on certain inner forms. Different to the analogous situation for $Gl(2)$, where this is clearly established by the Jacquet-Langlands lift, only partial results are known for Siegel modular forms. For instance, the existence of L-series for algebraic modular forms, satisfying all the expected properties, would be achieved with establishing this lift. But, seen the other way round, one could also try to construct such L-series a priori and then attempt to solve the remaining question by converse theorems. To establish the connection between Siegel modular forms and algebraic modular forms should be important not only for analytic aspects, say, concerning L-series, but even more significantly for some arithmetic questions like congruences between modular forms and the mod p-geometry. It seems to me that certain unexpected new phenomena might occur at various stages. I would like to formulate this topic as one of the important desiderata in order to probe deeper into the arithmetic of Siegel modular forms.

There are several approaches to attack the problem. One obvious strategy might be to use the Selberg trace formula or converse theorems, but there are different and more arithmetic options, for instance in the spirit of p-adic uniformation a la Cerednik-Drinfeld. The underlying question, on the other hand, is furthermore connected to the study of theta lifts relating these things to half integral modular forms a la Shimura and Waldspurger. If one believes in the analogy to modular curves, then some of the really subtle arithmetic information should be also encoded in here. Yet, although related, this is just another theme of interest. All in all, the questions centering around these topics are interesting enough from their own point of view, while they are also clearly related to the task of numerically providing something like the analog of Cremona's tables.

Neil: Let F be a genus-2 cuspidal Hecke eigenform of weight 3 and paramodular level N, in the minus space. For a prime p not dividing N, let the Euler factor in the spinor L-function of F be $f_p(p^{-s})^{-1}$. A good way to find examples and the $f_p(X)$ numerically appears to be using algebraic modular forms for the orthogonal groups of quinary quadratic forms. There is some data already (for some prime levels) in the Ph.D. thesis of Voight's student Hein [28, Appendix B]. It would be very nice to see an example of a CY 3-fold whose Frobenius eigenvalues appear to match some computed Hecke eigenvalues. If, for such an example, one then found an actual Siegel modular form, rather than just an algebraic modular form, so as to have an associated Galois representation, then maybe following the method applied by Brumer et.al. to abelian surfaces [6], one could even prove modularity.

Various possible congruences of Hecke eigenvalues for F (modulo a prime divisor q in a sufficiently large coefficient field) include the following:

(1) $f_p(X) \equiv (1-pX)(1-p^2X)(1-a_pX+p^3X^2) \pmod{q}$, where $g = \sum a_n q^n$ is a normalised newform of weight 4 and level N. (Congruence of Hecke eigenvalues between F and a Gritsenko lift.) Given $f_p(X)$ for several p, it is easy to check for this experimentally by factorising the $f_p(1/p)$ and looking for q as a common divisor, then looking in LMFDB for a g that fits. Examples visible in Hein's data include $(N,q) = (61,43)$,$(73,13)$,$(89,29)$,$(113,13)$,$(167,23)$,$(173,7)$ and $(197,13)$.

Such a congruence implies reducibility of an associated 4-dimensional mod q Galois representation. According to the Bloch-Kato conjecture, q (as the order of an element in a Selmer group arising from an extension of mod q Galois representations) should divide the numerator of a canonically normalised algebraic part of $L(g,3)$ (or strictly-speaking the incomplete $L_N(g,3)$). This can be confirmed in any given case by exact computation with the Magma command LRatio. A recent preprint of Jim Brown and Huixi Li (Congruence primes for automorphic forms on symplectic groups, http://jim-brown.oxycreates.org/research.html) deals with the converse. Given g of square-free level N and weight $2k-2$, with minus sign in the functional equation of its Hecke L-function, if q divides a normalised $L(g,k)$ then, under weak technical conditions, they prove a congruence of Hecke eigenvalues between the Gritsenko lift (paramodular Saito-Kurokawa lift) of g and a non-lift Siegel Hecke eigenform of weight k and paramodular level N. They also give an application to the Bloch-Kato conjecture. See their Theorems 6.9 and 7.4. The case $k=3$ is relevant here.

Also, q^2 (with q as the order of a "global torsion element") should divide the denominator of a suitable algebraic part of the central value $L(\mathrm{spin}, F, 2)$. To cancel an unknown Deligne period, we should look for q^2 in the numerator of the ratio $L(\mathrm{spin}, F, \chi_D, 2)/L(\mathrm{spin}, F, 2)$, where χ_D is a real quadratic character and $\left(\frac{D}{N}\right) = 1$. These twisted central spinor L-values appear in a generalisation of Böcherer's conjecture, due to Ryan and Tornaria [37, Conjecture B](ArXiv:1206:0072v1, 2018). As explained in a beautiful observation, their Proposition 5.1, the divisibility we seek can be deduced from their Conjecture B if one has a congruence $\pmod{q}$ of *Fourier coefficients* between F and a Gritsenko lift (not necessarily a Hecke eigenform), of the type appearing in [36, §8] (Poor and Yuen), where paramodular forms are constructed using theta blocks. Moreover, the experimental congruence of Hecke eigenvalues can easily be *proved* from such a congruence of Fourier coefficients. Thus at least the examples $(N,q) = (61,43), (73,13)$ are proved.

(2) $f_p(X) \equiv (1-X)(1-p^3X)(1-pb_pX+p^3X^2) \pmod{q}$, where $h = \sum b_n q^n$ is a normalised newform of weight 2 and level N. Given $f_p(X)$ for several p, it is easy to check for this experimentally by factorising the $f_p(1)$ and looking for q as a common divisor. Experimental examples visible in Hein's data include $(N,q) = (61,19)$,$(73,3)$, $(79,2)$, $(89,5)$, $(113,2)$. No example of such a congruence has actually been proved. Perhaps it would be possible in some space of algebraic modular forms. If $f = \sum c_n q^n$ is a normalised newform of weight 4 and level ℓ such that $c_p \equiv 1+p^3 \pmod{q}$ for all $p \neq \ell$ (e.g. one can do this with $\ell = 37$ when $q = 19$), then the Bloch-Kato conjecture predicts that q (maybe sufficiently large) divides the numerator of an algebraic part of the incomplete L-value $L_{\ell N}(f \otimes h, 3)$. In accord with this, q divides the missing Euler factor at 61, as a consequence of $c_p \equiv 1+p^3 \pmod{q}$ with $p = 61$ (or, more generally, dividing N). There doesn't seem to be any L-function where one would expect to see a q in the denominator as a result of the congruence.

Gonzalo:

Here's a short description of the method for computing algebraic modular forms for the orthogonal group:

The basic idea is to pick a suitable genus of positive definite quinary quadratic form. Associated to this genus is a certain space of algebraic modular forms, and one can use neighbouring lattice methods to construct the Hecke operators. This is a generalization of a method of [4] in the case of ternary quadratic forms (see [16], [28], [32]). The spaces thus constructed contain lifts of modular forms for GL(2), however it was noted by Hein-Ladd-Tornara that it's easy to compute the kernel of a theta map which will contain all the non-lifts in the space. Note also that, as in Birch method, this is expected to compute only forms with funcional equation with sign $+1$, however this difficulty can be overcome using characters of the spinor norm (this has been worked out in the case of ternary quadratic forms by Tornaria, Rama, Hein-Tornaria-Voight, and it should work the same for quinary quadratic forms).

Cris, David and Jerry:

General existence results for weight three nonlift paramodular cusp forms of prime level are due to Ibukiyama [29], who gave dimension formulae for weight three paramodular cusp forms of prime level, $S_3(K(p))$. In conjunction with the known dimensions of spaces of Jacobi forms, one can see that the first weight three nonlifts with rational eigenvalues occur at levels $p = 61$, 73, and 79. These same levels had been identified in the work of Ash, Gunnells, and McConnell [2] as occurring in $H^5(\Gamma_0(N), \mathbb{C})$; the congruence subgroup $\Gamma_0(N) \subseteq \mathrm{SL}_4(\mathbb{Z})$ is defined by having a bottom row in $(N\mathbb{Z}, N\mathbb{Z}, N\mathbb{Z}, \mathbb{Z})$. There is still no known mapping from nonlift paramodular cuspidal eigennewforms with rational eigenvalues into the cohomology space $H^5(\Gamma_0(N), \mathbb{C})$, but perhaps one can be constructed using the orthogonal point of view. These paramodular cusp forms for $p \in \{61, 73, 79\}$ were directly constructed by Poor and Yuen in [36]. The eigenforms were constructed there as rational functions of Gritsenko lifts of theta blocks. Using this construction, the 2-, 3-, and 5-Euler factors were computed, integral Fourier expansions of content one were given, and congruences modulo ℓ for the Fourier expansions of these nonlift eigenforms to those of Gritsenko lifts were identified in each case. For $p = 61$, we have $\ell = 43$; for $p = 73$, we have $\ell = 3, 13$; and for $p = 79$, we have $\ell = 2$. In view of the results in [26], it is clear that these same constructions may be written as a sum of Gritsenko lifts and a Borcherds product.

Borcherds products have turned out to be a very useful tool for the construction of paramodular cusp forms. Indeed, there is an algorithm on the arXiv [35] for classifying all Borcherds products in all spaces $S_k(K(N))$. The efficient implementation of this algorithm relies on a good knowledge of a determining number of Fourier–Jacobi coefficients for a given $S_k(K(N))$, and the algorithm has already been successfully used in [3] and in [33]. Even very basic assertions, such as knowing that a certain paramodular form is an eigenform, hinge on a rigorous spanning set for $S_k(K(N))$. To this end, upper and lower bounds for $\dim S_k(K(N))$ must be computed separately. For squarefree N there is a dimension formula, due to Ibukiyama and Kitayama [30]. For N not squarefree, the upper bounds are approachable via the method of *Jacobi restriction* [31, 5, 34], which classifies possible initial Fourier–Jacobi expansions of paramodular cusp forms. Lower bounds, on the other hand, require the construction of paramodular forms by techniques such as Borcherds products and Hecke spreading of Borcherds products, compare [34].

Once a relevant space of paramodular forms has been spanned, typically because there is a matching arithmetic object as a candidate for modularity, one can broach the separate question of computing enough Euler factors to rigorously prove this modularity. In weight two, such a strategy has recently had success in [6] for proving examples of modularity for typical abelian surfaces defined over $\mathbb{Q}$. The number of Euler factors that can be computed is sensitive to the manner of construction of the paramodular eigenform, but good results can be expected for paramodular eigenforms which are constructed as rational functions of Gritsenko lifts of theta blocks. One natural arithmetic candidate for modularity in the weight three case is given by the class of hypergeometric motives.

Dave Roberts sent Poor, Shurman, and Yuen (henceforth PSY) some hypergeometric motives with motivic Galois group GSp(4). One, for example, had conductor $N = 257$, and PSY did find a new form with rational eigenvalues in $S_3(K(257))$, whose 2-Euler factor in the arithmetic normalization was

$$1 + x + 6x^2 + 8x^3 + 64x^4.$$

This matched the 2-Euler factor that Dave Roberts had. The paramodular Atkin–Lehner sign here was -1. Such hypergeometric motives are good candidates for examples of rigorous modularity proofs following the pattern of [6]. Indeed, the theory of Galois representations is better understood in weight three than in weight two, and the existence of dimension formulae for prime level (and at least conjecturally for squarefree level) makes the weight three case look more approachable than the weight two case. PSY have recently written a number of new programs aimed at gathering computational data, both rigorous and heuristic, about weight three paramodular cusp forms. However, the dimensions of the weight three spaces grow more quickly than in weight two.

Also, Henri Cohen, in his *Computing L-functions: A survey* [11], says that the Dwork quintic pencil

$$x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 - 5\psi x_1x_2x_3x_4x_5, \qquad (\psi \in \mathbb{Q})$$

can give a hypergeometric motive of conductor $N = 525$, which should be modular with respect to a paramodular newform in $S_3(K(N))$. So far, however, no arithmetic geometer has presented any Euler factors for this case. Of course, the level $N = 525$ is not squarefree, but such examples have in principle been dealt with in [33], where $N = 16$ was successfully considered.

Due the start-up costs of beginning a computation, the best manner in which to proceed is for PSY to respond to motives with known conductors and Euler factors, trying to locate paramodular newforms that match them. If anyone wants to send PSY such a target, we will start working on that case to provide at least heuristic information.

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